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Relational Data Graphs and Some Properties of Them

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Data graph $\Gamma = (C, \Lambda)$ was proposed by A. L. Rosenberg as a mathematical description for data structure. In this paper, relational data graph $\mathbf{\Gamma} = (C, R)$ is newly defined to describe and investigate more general structure represented by a generalized directed graph in which more than two equilabeled edges emanate from a node. Each element of R is a relation rather than a function on the set C of data cells. Owing to the relationality of $r \in R$, in relational data graphs, a set of data items can be obtained by one retrieval procedure. The authors make various formulations, which would offer effective schemes for manipulating data structures. By studying structures of a relation $r \in R$, the authors develop many properties characteristic to them.

1. INTRODUCTION

Given a problem and its associated data, by analyzing the semantic structure of the problem, then imposing the problem-oriented structure on the data, we can construct an efficient procedure for solving the problem.

Though even in a case of a simple problem, because of the many criteria conflicting with each other, it is generally difficult to obtain the optimum structuring of the data.

However once a data structure is established according to some criteria and implemented in a computer, many important properties of the data structure, on the computational stage, become independent of the contents of data items.

Instead of the above-mentioned problem-oriented approach to the data structuring, studying properties of data structures, whose analysis depends only on their forms themselves, would be an effective approach. Investigating algebraic and graph-theoretical properties of the various structures underlying data structures, we can expect to discover the structures on which many fundamental manipulations can be applied effectively.

Much research has been done in concern with such morphological formulation of data structures; for example, Childs [8], Rosenberg [2–5], Fleck [6], Turski [7]. Among such excellent works, "Data Graph Theory," developed by A.L. Rosenberg, is a very enlightening one, in which he proposed a model simple enough to be treated mathematically. A data graph is obtained from a data structure by masking out the specific data items at the nodes of the structure and concentrating only on the linkages in the structure.

Linkages denoting various "relations" among data items are partial functions λ's on C (the set of data cells). Data graph is defined in terms of these functions. Two notions arising

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in data graph realization have been isolated, namely relative addressing and relocatability, which can be studied in terms of the structure of the data graphs involved. In [2], these two notions are precisely formulated and those data graphs are characterized to which these two notions are applicable. In [3–5], the properties of those data graphs are investigated in detail.

In his formulation of data graph however, owing to the functionality of λ's, at most one item is related to some item by each of λ's. This makes it inevitable that only one data item is obtained by one retrieval procedure. This would be a vital limitation when the size of data structures become large and fast processing is demanded.

In this paper, relational data graph $\Gamma = (C, R)$ is newly defined to describe and investigate more general structure as is represented in Fig. 1 in which more than two equilabeled edges emanate from a node. Each element of R is a relation rather than a

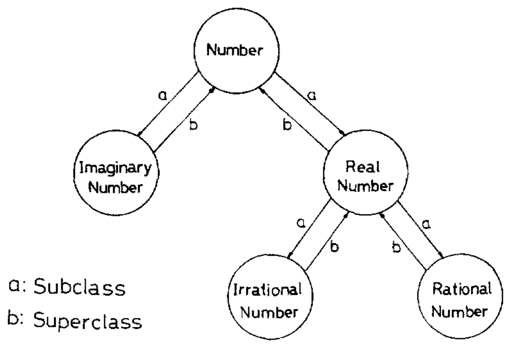


FIG. 1. A representation of a data structure by a generalized directed graph.

function on the set C of data cells. The authors introduce "block partitionable relational data graph" which is mainly discussed in this research. In a block partitionable relational data graph, owing to the relationality of $r \in R$, a set of data items called a "block" can be obtained by one retrieval procedure. So a higher rank data such as a set of blocks can be also successfully treated in our new model.

The authors make various formulations such as a "root block" (a generalization of "root" in [2]) or a "skeleton structure" which would offer effective schemes for manipulating data structures.

It is shown that some of the results in [2–5] are naturally extended in our new model, and by investigating structures of a relation $r \in R$, the authors develop various features and properties characteristic to the class of relational data graphs.

2. Relational Data Graphs and Block Partitionability

In this section, first, a relational data graph is specified. Then, a block partitionable relational data graph is introduced and its several properties are studied. On the block partitionable relational data graphs, we formulate a class of relational data graphs which admit a "root block," and investigate various properties of them. It is stated in the later section that relational data graphs with root blocks do enjoy an effective realization scheme on a memory space of a computer.

First, we establish the following notational conventions.

Notation. Some of the symbols used in [2] may be also employed in this paper.

Let R be a set of binary relations on a set C; for each $r \in R$, $r \subseteq C \times C$.

R^τ is the monoid of relations generated from R under compositions of relations. Each binary relation $\xi \in R^\tau$ is viewed as a function $\xi: C \to 2^C$ (the power set of C), and for each $\xi \in R^\tau$, $c\xi = \{c' \in C \mid \langle c, c' \rangle \in \xi\}$.

$\nabla_R(c)$ is a set of relations and compositions of relations defined on $c \in C$, that is $\nabla_R(c) = \{\xi \in R^\tau \mid ^\exists c' \in C, \langle c, c' \rangle \in \xi\}$.

Definition 1. A relational data graph (rdg for short) is specified as an ordered pair $\mathbf{\Gamma} = (C, R)$, where

(i) C is a countable set of data cells;

(ii) R is a finite set of relations defined on C;

(iii) For all c, $d \in C$, there exists a relation $\xi \in R^\tau$ such that $d \in c\xi$, namely $\mathbf{\Gamma}$ is represented by a strongly connected directed graph.

Rosenberg's definition of a data graph is exactly equal to the definition of an rdg restricting relations to functions in (ii).

Definition 2. Let $\mathbf{\Gamma} = (C, R)$ be an rdg. If the following condition obtains, then $\{b_0\}$ is called a *base block* of $\mathbf{\Gamma}$, and $\mathbf{\Gamma}$ is called a block partitionable rdg (bprdg for short). For any $\xi, \eta \in \nabla_R(b_0)$,

$$b_0\xi \cap b_0\eta \neq \varnothing \Rightarrow b_0\xi = b_0\eta.$$

Note that base blocks are singleton sets.

The set of base blocks of $\mathbf{\Gamma}$ is denoted by $\mathbb{B}_{\mathbf{\Gamma}}$.

Example 1. Figure 2 is an example of a bprdg where $\mathbb{B}_{\mathbf{\Gamma}} = \{\{①\}, \{⑨\}\}$.

Definition 3. Let $\mathbf{\Gamma} = (C, R)$ be a bprdg and $\{b_0\} \in \mathbb{B}_{\mathbf{\Gamma}}$. Then the relation $\simeq_{b_0}$ on C is defined as follows. For all $c_1, c_2 \in C$,

$$c_1 \simeq_{b_0} c_2 \Leftrightarrow ^\exists \xi \in \nabla_R(b_0), \{c_1, c_2\} \subseteq b_0\xi.$$

Proposition 1. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg *and* $\{b_0\} \in \mathbb{B}_{\mathbf{\Gamma}}$. *Then, the relation* $\simeq_{b_0}$ *is an equivalence relation on* C.

Proof. That $\simeq_{b_0}$ is reflexive and symmetric is obvious from the definition and the strong connectivity of $\mathbf{\Gamma}$. We show the transitivity of $\simeq_{b_0}$. For each $c_1, c_2, c_3 \in C$, let $c_1 \simeq_{b_0} c_2$ and $c_2 \simeq_{b_0} c_3$. Then, from the definition of $\simeq_{b_0}$, there exist $\xi, \eta \in \nabla_R(b_0)$ such that $\{c_1, c_2\} \subseteq b_0\eta$ and $\{c_2, c_3\} \subseteq b_0\eta$. Hence, $b_0\xi \cap b_0\eta \neq \varnothing$.

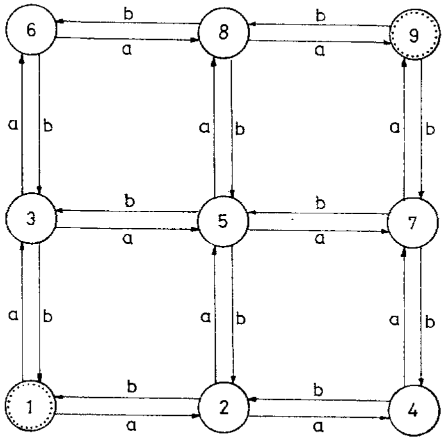


FIG. 2. A block partitionable rdg.

Since, $\{b_0\}$ is a base block of $\mathbf{\Gamma}$, $b_0\xi = b_0\eta$. Therefore $\{c_1, c_3\} \subseteq b_0\xi$, so that $c_1 \simeq_{b_0} c_3$. That $\simeq_{b_0}$ is an equivalence relation on C is now shown. ∎

From the above proposition, we can see that C is partitioned by the equivalence relation $\simeq_{b_0}$. Each equivalence class is called a *block of* $\mathbf{\Gamma}$ *induced by the base block* $\{b_0\}$.

Let $\mathscr{B}_{\mathbf{\Gamma}}[b_0]$ denote the set of blocks of $\mathbf{\Gamma}$ induced by $\{b_0\}$, then $\mathscr{B}_{\mathbf{\Gamma}}[b_0] = \{B \mid B = b_0\xi, \xi \in \nabla_R(b_0)\}$.

EXAMPLE 2. For the bprdg $\mathbf{\Gamma}$ in Fig. 2, $\mathscr{B}_{\mathbf{\Gamma}}[①] = \mathscr{B}_{\mathbf{\Gamma}}[⑨] = \{\{①\}, \{②, ③\}, \{④, ⑤, ⑥\}, \{⑦, ⑧\}, \{⑨\}\}$.

Base block $\{b_0\}$ is an entry block of $\mathbf{\Gamma}$. Accessing to each block of $\mathbf{\Gamma}$ can be successfully accomplished by starting from the base block cell b_0.

Some of the properties of bprdg's are developed, which result in Theorem 3.

LEMMA 2. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg. *For any cell* $c \in C$, *any base block* $\{b_0\} \in \mathbb{B}_{\mathbf{\Gamma}}$, *and* $\xi \in \nabla_R(c)$,

$$b_0 \in c\xi \Rightarrow c\xi = \{b_0\}.$$

Proof. From the strong connectivity of $\mathbf{\Gamma}$, there exists $\eta \in \nabla_R(b_0)$ such that $c \in b_0\eta$. Say, there exists $d \in c\xi$ such that $d \neq b_0$. Then $b_0\eta\xi \supseteq \{d, b_0\}$, but since $1_C \in R^\tau$ is defined at every cell in C, $b_0 1_C = \{b_0\}$, so that $b_0\eta\xi \cap b_0 1_C \neq \varnothing$ and $b_0\eta\xi \neq b_0 1_C$. This contradicts that $\{b_0\}$ is a base block of $\mathbf{\Gamma}$, that is, there are no $d \in c\xi$ such that $d \neq b_0$. Hence, $c\xi = \{b_0\}$. ∎

THEOREM 3. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg. *For any two base blocks* $\{b_1\}, \{b_2\} \in \mathbb{B}_{\mathbf{\Gamma}}$,

$$\mathscr{B}_{\mathbf{\Gamma}}[b_1] = \mathscr{B}_{\mathbf{\Gamma}}[b_2].$$

Proof. Let $B \in \mathscr{B}_{\mathbf{\Gamma}}[b_1]$ and $b_1\xi = B(\xi \in \nabla_R(b_1))$. For $\zeta \in \nabla_R(b_1)$, we assume that $b_2 \in b_1\zeta$. Then from Lemma 2, $b_1\zeta = \{b_2\}$. For any $c \in B$, there is an $\eta \in \nabla_R(b_2)$ such that

$c \in b_2\eta$ from the strong connectedness of Γ. Hence, $b_1\xi \cap b_1\zeta\eta \neq \varnothing$. Since $\{b_1\}$ is a base block of Γ, $b_1\xi = b_1\zeta\eta$ from Definition 2, namely $b_1\xi = b_2\eta$. Since $b_2\eta \in \mathscr{B}_\Gamma[b_2]$, $b_1\xi = B \in \mathscr{B}_\Gamma[b_2]$. Conversely, for any block $B \in \mathscr{B}_\Gamma[b_2]$, $B \in \mathscr{B}_\Gamma[b_1]$ is shown in the same way. Thus $\mathscr{B}_\Gamma[b_1] = \mathscr{B}_\Gamma[b_2]$ follows. ∎

The above theorem implies that the partitioning of C by the equivalence relation $\simeq_{b_0}$ gives the same set of blocks, which is independent of our choice of a base block $\{b_0\} \in \mathbb{B}_\Gamma$. This independence permits us from now on to denote the set of blocks of a bprdg simply as $\mathscr{B}_\Gamma$ without b_0.

DEFINITION 4. Let $\Gamma = (C, R)$ be a bprdg. The *blocking mapping* $\gamma: C \to \mathscr{B}_\Gamma$ is defined as follows. For each $B \in \mathscr{B}_\Gamma$,

$$c\gamma = B \Leftrightarrow c \in B.$$

By the blocking mapping of Γ, each cell $c \in C$ is alloted to the block which contains it.

DEFINITION 5. Let $\Gamma = (C, R)$ be a bprdg. If there exists $\{c_0\} \in \mathbb{B}_\Gamma$ such that for all $\xi, \eta \in \nabla_R(c_0)$,

$$c_0\xi = c_0\eta \Rightarrow \xi = \eta \tag{1}$$

is satisfied, $\{c_0\}$ is called a *root block* of Γ.

EXAMPLE 3. Figure 3A is an rdg which admits a root block $\{①\}$ and Fig. 3B is an rdg which admits two root blocks $\{①\}$ and $\{②\}$.

Let $\Gamma = (C, R)$ be an rdg which admits a root block $\{c_0\}$. For an arbitrary block B of Γ, there exists a relation $\xi \in \nabla_R(c_0)$ such that $B = c_0\xi$. From (1) we can see that all of the relations $\eta \in \nabla_R(c_0)$ such that $B = c_0\eta$ are exactly equal (as binary relations) to ξ. Hence a unique relation from the root block cell c_0 can be assigned to each block of Γ. This unique relation is designated as the address of the block.

This notion of a "root block" is a generalization of "root" in [2].

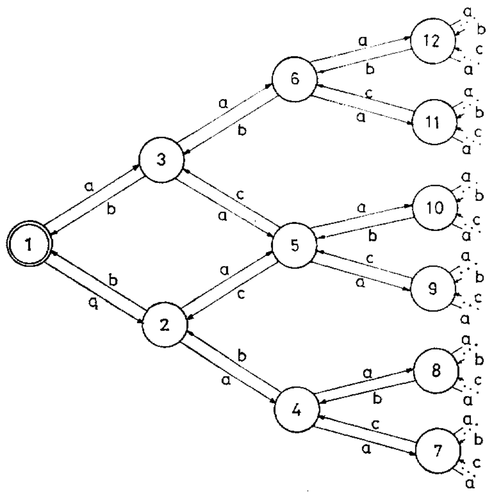
Several properties and features of an rdg with root blocks are provided by investigating the structures of relations in R^τ.

LEMMA 4. *Let* $\Gamma = (C, R)$ *be an* rdg *with a root block* $\{c_0\}$. *If* $c_0 \in c_0\xi$[1] *for any* $\xi \in \nabla_R(c_0)$, $\xi = 1_C$.

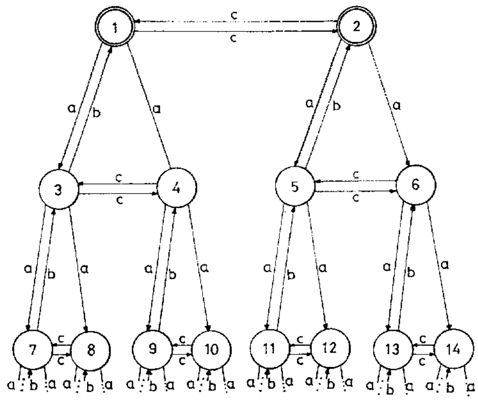
Proof. Since $c_0 1_C = \{c_0\}$, $c_0 1_C \cap c_0\xi \neq \varnothing$. By the block partitionability of Γ, $c_0 1_C = c_0\xi$. Since $\{c_0\}$ is a root block of Γ, $\xi = 1_C$ from (1). ∎

LEMMA 5. *Let* $\Gamma = (C, R)$ *be an* rdg *with a root block* $\{c_0\}$. *Each* $\xi \in \nabla_R(c_0)$ *is total. Here "a relation* $\xi \in R^\tau$ *is total" implies that* $\xi \in \nabla_R(c)$ *for every cell* $c \in C$; *that is, when viewed as a function,* ξ *is total on* C.

[1] From Lemma 2, in fact, $c_0\xi = \{c_0\}$.



(A)



(B)

FIG. 3. Two examples of rdg with root blocks.

Proof. For each $c \in c_0\xi$, there exists an $\eta \in \nabla_R(c)$ such that $c_0 \in c\eta$ because of the strong connectivity of Γ. Therefore $c_0 \in c_0\xi\eta$ holds. From the above lemma, $\xi\eta = 1_C$ is obtained. The totality of 1_C ensures that ξ is total on C.

Now let $\mathscr{R}_\Gamma$ denote the set of root blocks of an rdg Γ. It is verified that an rdg with root blocks has the characteristic property exhibited in the following theorem.

THEOREM 6. *Let $\Gamma = (C, R)$ be an* rdg *with root blocks. For any $\{c_1\}, \{c_2\} \in \mathscr{R}_\Gamma$, every $\xi \in \nabla_R(c_1)$ such that $c_2 \in c_1\xi$*[2] *is a function from C to C.*

[2] From Lemma 2, in fact $c_1\xi = \{c_2\}$.

Proof. From the strong connectivity of $\mathbf{\Gamma}$, there exists $\eta \in \nabla_R(c_2)$ such that $c_1 \in c_2\eta$. Then $c_1 \in c_1\xi\eta$ holds. Since $\{c_1\}$ is a root block of $\mathbf{\Gamma}$, $\xi\eta = 1_C$ from Lemma 4. Similarly, $\eta\xi = 1_C$ since $\{c_2\}$ is also a root block of $\mathbf{\Gamma}$. Now assume the existence of c, c_3, $c_4 \in C$ such that $c_3 \in c\xi$ and $c_4 \in c\xi$ but $c_3 \neq c_4$. Since $\xi\eta = 1_C$, for all $c' \in c\xi$ we have $c'\eta = \{c\}$. Hence, $c_3 \in c'\eta\xi$ and $c_4 \in c'\eta\xi$, while $c_3 \neq c_4$, and this contradicts $\eta\xi = 1_C$. Therefore for each $c \in C$, there is at most one element in $c\xi$, that is to say ξ is a function on C. ∎

For example, in Fig. 3B the relation $c \in R$ between the two root blocks {①} and {②} is a function. From Theorem 7 or from Lemma 2, the next corollary is obtained.

COROLLARY 7. *If there is a relation $r \in R$ which is not a function, there exists no* rdg $\mathbf{\Gamma} = (C, R)$ *such that every cell is a root block cell.*

3. SKELETON STRUCTURES OF RDG'S

In a bprdg $\mathbf{\Gamma} = (C, R)$ defined in the previous section, not only the structure among the cells (C-structure), but also the structure among the blocks ($\mathscr{B}_\mathbf{\Gamma}$-structure) can be described. One of our next concerns is to extract the $\mathscr{B}_\mathbf{\Gamma}$-structure from a bprdg $\mathbf{\Gamma}$. To specify the $\mathscr{B}_\mathbf{\Gamma}$-structure separately from the C-structure would contribute to make the processing of each block itself easier.

In this section, the authors provide the skeleton mapping $h = \langle \epsilon, \kappa \rangle$ of a bprdg $\mathbf{\Gamma} = (C, R)$. The mapping h reveals the skeleton structure of $\mathbf{\Gamma}$, that is $h(\mathbf{\Gamma}) = (S, R')$. This skeleton structure $h(\mathbf{\Gamma})$ serves itself as an index of $\mathscr{B}_\mathbf{\Gamma}$-structure of Γ. ϵ maps each cell $c \in C$ to a single cell $s \in S$ which denotes the block containing the cell, and κ maps each relation $r \in R$ to a function r' on S. It is proved that the existence of a root block in $\mathbf{\Gamma}$ is preserved under the mapping h. In addition, for a bprdg $\mathbf{\Gamma}$ a condition is provided which ensures the existence of roots in $h(\mathbf{\Gamma})$.

DEFINITION 6. Let $\mathbf{\Gamma} = (C, R)$ be a bprdg. The *skeleton mapping* of $\mathbf{\Gamma}$ is a pair of mappings defined as follows. S is an arbitrary set of cells such that $\#(\mathscr{B}) = \#(S)$.

$$h = \langle \epsilon, \kappa \rangle;$$

where $\epsilon = \gamma u$,

$\gamma\colon C \to \mathscr{B}_\mathbf{\Gamma}$ (blocking mapping),

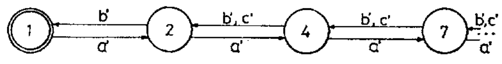
$u\colon \mathscr{B}_\mathbf{\Gamma} \to S$ is an arbitrary one-to-one total function,

$\kappa\colon R \to R' = \{r' \mid r \in R\}$ is a total function and for any $r \in R$, $r' \in R'$ is specified according to the following rule,

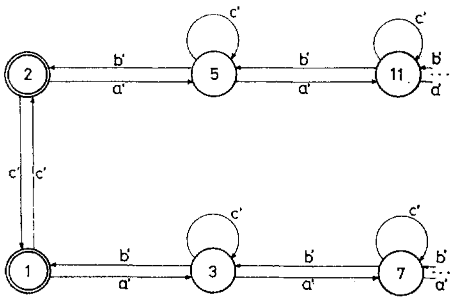
$$r' = \{\langle c_1\epsilon, c_2\epsilon \rangle \mid \langle c_1, c_2 \rangle \in r\}. \tag{2}$$

$h(\mathbf{\Gamma})$ is called the *skeleton structure* of $\mathbf{\Gamma}$.

EXAMPLE 4. The skeleton structures of Fig. 3A and 3B are depicted in Fig. 4A′ and 4B′, respectively.



(A′)



(B′)

FIG. 4. The skeleton structures of rdg's depicted in Fig. 3.

Now for each $\xi = r_1 r_2 \cdots r_n \in R^\tau$ (each r_i is contained in R), let ξ' be as follows

$$\xi' = r_1' r_2' \cdots r_n' \qquad (r_i \in R).$$

The mapping ϵ is generally a many-to-one function from C to S. From the definition of the block partitionability, $c_1\gamma = c_2\gamma$ holds for each $c \in C$, $\xi \in \nabla_R(c)$, and $c_1, c_2 \in c\xi$. Then from the one-to-oneness of u, $c_1\epsilon = c_2\epsilon$ is obtained. Therefore the notation $(c\xi)\epsilon$ is permitted and it in fact denotes $d\epsilon$ for arbitrary cell $d \in c\xi$. Then, from Eq. (2), for each $c \in C$ and $r \in \nabla_R(c)$,

$$(c\epsilon)\, r' = (cr)\epsilon \tag{3}$$

is obtained. Equation (3) insures that $r' \in R'$ is a function on C'. And the strong connectedness of $h(\boldsymbol{\Gamma})$, the fact that for each $s_1, s_2 \in S$, there exists $\xi' \in R'^\tau$ such that $s_1\xi' = s_2$, is guaranteed from the strong connectedness of $\boldsymbol{\Gamma}$ and Definition 6. Hence, $h(\boldsymbol{\Gamma}) = (C\epsilon, R\kappa)$ specifies an rdg $\boldsymbol{\Gamma}' = (S, R')$ where each $r' \in R'$ is a function on S, and owing to the functionality of $r' \in R'$, it follows that $\boldsymbol{\Gamma}'$ is block partitionable. $h(\boldsymbol{\Gamma})$ is a data graph in the sense of [2].

Equation (3) is now extended in the following proposition.

PROPOSITION 8. *Let $\boldsymbol{\Gamma}$ be a* bprdg. *For each $c \in C$ and each $\xi \in \nabla_R(c)$,*

$$(c\epsilon)\, \xi' = (c\xi)\epsilon. \tag{4}$$

Proof. Let $\xi = r_1 r_2 \cdots r_m$ $(r_i \in R)$ and $\xi_j = r_1 r_2 \cdots r_j$ $(1 \leqslant j \leqslant m-1)$. There exists $d \in c\xi_j$ such that $r_{j+1} \in \nabla_R(d)$ for each $1 \leqslant j \leqslant m-1$. By Eq. (3), $(d\epsilon)\, r'_{j+1} = (dr_{j+1})\epsilon$. And $d \in c\xi_j$ yields $d\epsilon = (c\xi_j)\epsilon$ and $dr_{j+1} \subseteq c\xi_j r_{j+1}$ yields $(dr_{j+1})\epsilon = (c\xi_j r_{j+1})\epsilon$, so that $(c\xi_j)\epsilon r'_{j+1} = (c\xi_j r_{j+1})\epsilon$. Hence, $(c\epsilon)\xi' = (c\epsilon) r_1' r_2' \cdots r_m' = (cr_1)\epsilon r_2' \cdots r_m' = (cr_1 r_2 \cdots r_m)\epsilon = (c\xi)\epsilon$. This completes the proof. ▌

Here let the domain of κ extend from R to R^τ as follows.

$$\text{For each } \xi \in R^\tau, \qquad \xi\kappa = \xi'.$$

The well-definedness of this extension will be made sure as follows. For all $r_1, ..., r_m, r_{m+1}, ..., r_n \in R$, let $r_1 r_2 \cdots r_m = r_{m+1} r_{m+2} \cdots r_n$. Then for arbitrary cell c in the domain of $r_1 \cdots r_m$ (or $r_{m+1} \cdots r_n$), $cr_1 \cdots r_m = cr_{m+1} \cdots r_n$, so $(cr_1 \cdots r_m)\epsilon = (cr_{m+1} \cdots r_n)\epsilon$. From Eq. (4), $(c\epsilon)\, r_1' \cdots r_m' = (c\epsilon)\, r_{m+1}' \cdots r_n'$. Since c is arbitrary, $r_1' \cdots r_m' = r_{m+1}' \cdots r_n'$ is obtained. Therefore κ is a total function from R^τ to R'^τ.

PROPOSITION 9. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg *and* $\{b_0\} \in \mathbb{B}_\Gamma$. *For* $\xi, \eta \in \nabla_R(b_0)$, *if* $(b_0\epsilon)\, \xi' = (b_0\epsilon)\, \eta'$, *then* $b_0\xi = b_0\eta$.

Proof. If $(b_0\epsilon)\, \xi' = (b_0\epsilon)\eta'$, by Proposition 8 $(b_0\xi)\epsilon = (b_0\eta)\epsilon$, namely, $(b_0\xi)\, \gamma u = (b_0\eta)\gamma u$ for $\epsilon = \gamma u$. Since u is one-to-one, $(b_0\xi)\gamma = (b_0\eta)\gamma$. b_0 is a base block cell, so $b_0\xi$, $b_0\eta \in \mathscr{B}_\Gamma$ therefore $(b_0\xi)\gamma = b_0\xi$ and $(b_0\eta)\gamma = b_0\eta$. Hence, $b_0\xi = b_0\eta$. ∎

PROPOSITION 10. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg *and* $h(\mathbf{\Gamma}) = (S, R')$. For any $s \in S$ *and* $\xi' \in R'^\tau$, *if* $\xi' \in \nabla_{R'}(s)$, *there exists* $c \in su^{-1}$ *such that* $\xi \in \nabla_R(c)$.

Proof. Immediate from Eq. (2). ∎

PROPOSITION 11. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg. *For* $\{b_0\} \in \mathbb{B}_\Gamma$ *and* $\xi' \in \nabla_{R'}(b_0')$, $\xi \in \nabla_R(b_0)$.

Proof. Immediate from Proposition 10, since the base block $\{b_0\}$ is a singleton set. ∎

If $h(\mathbf{\Gamma}) = (S, R')$ has a root block $\{s_0\}$, s_0 is simply referred to as a root of $h(\mathbf{\Gamma})$. Such $h(\mathbf{\Gamma})$ is a rooted data graph in the sense of [2]. Hereafter, $c\epsilon$ is often denoted as c'.

THEOREM 12. *If an* rdg $\mathbf{\Gamma} = (C, R)$ *has a root block* $\{c_0\}$, *then* $h(\mathbf{\Gamma})$ *has a root* c_0'.

Proof. For each $\xi', \eta' \in \nabla_R(c_0')$, let $c_0'\xi' = c_0'\eta'$. From Proposition 11, $\xi, \eta \in \nabla_R(c_0)$, so $c_0\xi = c_0\eta$ by Proposition 9. Then $\xi = \eta$, because $\{c_0\}$ is a root block of $\mathbf{\Gamma}$. So $\xi\kappa = \eta\kappa$, i.e., $\xi' = \eta'$. Hence, c_0' is a root of $h(\mathbf{\Gamma})$. ∎

EXAMPLE 5. Figure 3A has a root block {①}, while its skeleton structure, Fig. 4A′, has a root. Figure 3B has root blocks {①}, {②}, while its skeleton structure Fig. 4B′ has roots ① and ②.

It is made clear by the above theorem that the existence of a root block is preserved under the skeleton mapping h, but the existence of root blocks in $\mathbf{\Gamma}$ is not a necessary condition to ensure that $h(\mathbf{\Gamma})$ has roots. The following theorem affords a necessary and sufficient condition to guarantee the existence of roots in $h(\mathbf{\Gamma})$.

THEOREM 13. *Let* $\mathbf{\Gamma} = (C, R)$ *be a* bprdg. *The data graph* $h(\mathbf{\Gamma})$ *has a root if and only if there exists* $B_0 \in \mathscr{B}_\Gamma$ *such that for any* $c_1, c_2 \in B_0$ *and any* $\xi \in \nabla_R(c_1)$, $\eta \in \nabla_R(c_2)$,

$$(c_1\xi)\gamma = (c_2\eta)\gamma \Rightarrow \forall B \in \mathscr{B}_\Gamma,\ \exists d_1, d_2 \in B,\ (d_1\xi)\gamma = (d_2\eta)\gamma. \tag{5}$$

Proof. (i) First, assume that $h(\Gamma)$ has a root s_0 and $B_0 = s_0 u^{-1}$. For any c_1, $c_2 \in B_0$ and any $\xi \in \nabla_R(c_1)$, $\eta \in \nabla_R(c_2)$, let $(c_1\xi)\gamma = (c_2\eta)\gamma$. Then, $(c_1\xi)\epsilon = (c_2\eta)\epsilon$ applying the functionality of u and $\epsilon = \gamma u$. Since $\xi \in \nabla_R(c_1)$ and $\eta \in \nabla_R(c_2)$, by Proposition 8 $(c_1\epsilon)\xi' = (c_2\epsilon)\eta'$. c_1, $c_2 \in B_0$ implies $c_1\epsilon = c_2\epsilon = s_0$ and since s_0 is a root of $h(\Gamma)$, $\xi' = \eta'$. ξ', $\eta' \in \nabla_{R'}(s_0)$, therefore from Lemma 6, ξ' and η' are total on S, that is, for an arbitrary $s \in S$, $s\xi' = s\eta'$. Then from Proposition 10, d_1, $d_2 \in B = su^{-1}$ exist such that $\xi \in \nabla_R(d_1)$, $\eta \in \nabla_R(d_2)$.

Since for arbitrary $s \in S$, $s\xi' = s\eta'$, for such d_1, $d_2 \in B$ $(d_1\epsilon)\xi' = (d_2\epsilon)\eta'$. Then from Proposition 8, $(d_1\xi)\epsilon = (d_2\eta)\epsilon$. As u is one-to-one, $d_1\xi\gamma = d_2\eta\gamma$ is obtained. Hence (5) follows.

(ii) Conversely, let $B_0 \in \mathscr{B}_\Gamma$ exists such that for any c_1, $c_2 \in B_0$, any $\xi \in \nabla_R(c_1)$, $\eta \in \nabla_R(c_2)$, (5) holds. Let $B_0 u = s_0$ and for any ξ', $\eta' \in \nabla_{R'}(s_0)$, $s_0\xi' = s_0\eta'$. Then from Proposition 10, c_1, $c_2 \in B_0$ exist such that $\xi \in \nabla_R(c_1)$, $\eta \in \nabla_R(c_2)$. Since $c_1\epsilon = c_2\epsilon = s_0$, $(c_1\epsilon)\xi' = (c_2\epsilon)\eta'$. Proposition 8 implies that $(c_1\xi)\epsilon = (c_2\eta)\epsilon$, so $(c_1\xi)\gamma = (c_2\eta)\gamma$. Then from (5) for arbitrary $B \in \mathscr{B}_\Gamma$, there exist d_1, $d_2 \in B$ such that $(d_1\xi)\gamma = (d_2\eta)\gamma$. By Proposition 8, $(d_1\epsilon)\xi' = (d_2\epsilon)\eta'$ and $d_1\epsilon = d_2\epsilon = Bu$. Since B is an arbitrary block in $\mathscr{B}_\Gamma$, Bu is an arbitrary cell in S. Therefore $\xi' = \eta'$, and $h(\Gamma)$ has a root s_0.

Thus the theorem is proved. ▌

Condition (5) implies the following fact that if $c_1\xi$ and $c_2\eta$ are contained in the same block, then for an arbitrary block $B \in \mathscr{B}_\Gamma$, there exist d_1, d_2 in B such that $d_1\xi$ and $d_2\eta$ are contained in the same block.

EXAMPLE 6. There are no root blocks in Fig. 5, but its skeleton structure given in Fig. 6 has a root ①.

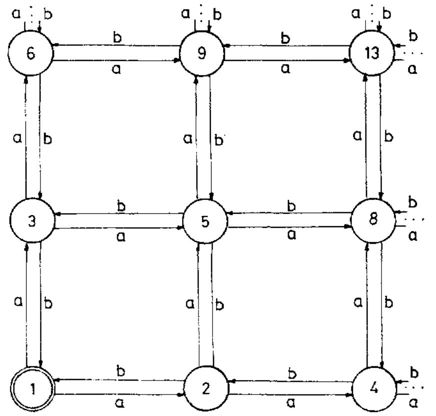


FIG. 5. rdg with no root blocks but its skeleton structure has a root.

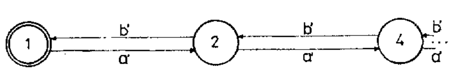


FIG. 6. The skeleton structure of the rdg depicted in Fig. 5.

4. Self-Embeddings of rdg's

In this section, the authors provide some classes of bprdg's in which every operation on $\mathbf{\Gamma}$ is also applicable recursively to its substructures. A bprdg $\mathbf{\Gamma}$ with a self-embedding θ (mapping from C to C) is defined. θ embeds $\mathbf{\Gamma}$ itself into its substructures. Two kinds of self-embeddabilities are provided and some of their properties are clarified. Here the property "θ_c-redundancy" is introduced, which was never discussed in the functional model (data graph).

Definition 7. A *self-embedding* of a bprdg $\mathbf{\Gamma} = (C, R)$ is a total injection (one-to-one into) $\theta: C \to C$, satisfying the condition that for an arbitrary $c \in C$ and $r \in R$,

$$cr \neq \phi^3 \Rightarrow (cr)\theta \subseteq (c\theta)r.$$

$\mathbf{\Gamma}$ is said to be *uniformly self-embeddable* if there is a $\{b_0\} \in \mathbb{B}_{\mathbf{\Gamma}}$ such that for all $c \in C$ there is a self-embedding θ_c of $\mathbf{\Gamma}$ with $b_0\theta_c = c$.

Example 7. Figure 5 given in Example 5 is a uniformly self-embeddable rdg. Figure 7 is also an example of a uniformly self-embeddable rdg. In Fig. 7, the next functions,

$$\theta_{③}{}^1 = \{\langle ⓪, ③\rangle, \langle ①, ⑥\rangle, \langle ②, ⑫\rangle, \langle ③, ⑬\rangle, ...\}$$
$$\theta_{③}{}^2 = \{\langle ⓪, ③\rangle, \langle ①, ⑦\rangle, \langle ②, ⑭\rangle, \langle ③, ⑮\rangle, ...\}$$

are both self-embeddings.

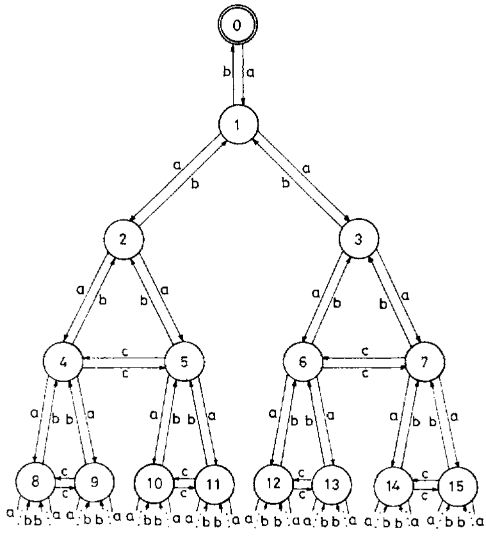


Fig. 7. A uniformly self-embeddable rdg.

[3] $cr \neq \varnothing \Leftrightarrow r \in \nabla_R(c)$.

Accessing cells $B\theta_{③}^{1}$ or $B\theta_{③}^{2}$ which starts from cell ③ can be accomplished by the same procedure which access a block $B \in \mathscr{B}_{\Gamma}$ starting from base block cell ⓪.

From the above example, when Γ is uniformly self-embeddable, for each $c \in C$, more than one θ_c may exist. But in the functional case, θ_c is uniquely specified for each $c \in C$ [2].

Now by a straightforward induction, we can extend the condition in Definition 7 from $r \in R$ to $\xi \in R^{\tau}$.

Intuitively, one can imagine a self-embedding as taking a copy of Γ and laying it over a second copy so that every node and edge of the first copy covers a corresponding element of the second. This is reflected in the assertion that $cr \neq \varnothing$ implies $(cr)\theta \subseteq (c\theta)r$. For example in Fig. 7, however if $\theta_{③}^{1}$ is chosen, many cells such as ⑦, ⑭, ⑮ and links such as $\langle$⑫, ⑬$\rangle$, $\langle$③, ⑦$\rangle$ fail to be covered by the first copy. Next we will specify such uncovered cells and links.

Let Γ be uniformly self-embeddable and b_0 satisfy the condition of uniformly self-embeddability in Definition 7. For the θ_c specified, let $C_{\theta_c} = \bigcup_{\xi \in \nabla_R(b_0)} (b_0\theta_c)\xi$, and for each $r \in R$, let $r\theta_c = \{\langle c_1\theta_c, c_2\theta_c\rangle \mid \langle c_1, c_2\rangle \in r\}$. Each cell in $C_{\theta_c} - C\theta_c$ is called a θ_c-*redundant cell.* And each link in $\bigcup_{r \in R} (r \mid C_{\theta_c} - r\theta_c)$ is called a θ_c-*redundant link.* Here $r \mid C_{\theta_c}$ is the restriction of r to C_{θ_c}.

THEOREM 14. *If a* bprdg $\Gamma = (C, R)$ *is uniformly self-embeddable, then* $h(\Gamma)$ *has a root.*

Proof. Since Γ is uniformly self-embeddable, there exists $\{b_0\} \in \mathbb{B}_{\Gamma}$ such that for all $c \in C$, there is a self-embedding θ_c of Γ with $b_0\theta_c = c$. On $h(\Gamma) = (S, R')$, for arbitrary ξ', $\eta' \in \nabla_{R'}(b')$, we assume $b_0'\xi' = b_0'\eta'$. Then, from Proposition 11, $\xi, \eta \in \nabla_R(b_0)$. From this and Proposition 9, $b_0\xi = b_0\eta$ holds, so $(b_0\xi)\gamma = (b_0\gamma)\eta$. Therefore from Theorem 13, it is sufficient to say that for any $c \in C$, $(c\xi)\gamma = (c\eta)\gamma$. First from the condition in Definition 7, for any $c \in C$, both $(b_0\xi)\theta_c \subseteq (b_0\theta_c)\xi = c\xi$, and $(b_0\eta)\theta_c \subseteq (b_0\theta_c)\eta = c\eta$ hold. $b_0\xi = b_0\eta$ and the functionality of θ_c result in that $(b_0\xi)\,\theta_c = (b_0\eta)\,\theta_c \neq \varnothing$. So $c\xi \cap c\eta \neq \varnothing$. Hence, $(c\xi)\gamma = (c\eta)\gamma$ and $b_0' = b_0\epsilon$ is a root of $h(\Gamma)$. ∎

EXAMPLE 8. The skeleton structure of Fig. 5 afforded in Fig. 6 has a root ①. And the skeleton structure of Fig. 7 has a root ①ϵ.

Next, strengthening Definition 7, we give another self-embedding in which both θ_c-redundant cells and θ_c-redundant links are precluded.

DEFINITION 8. A *self-isomorphic-embedding* of a bprdg $\Gamma = (C, R)$ is a total injection θ: $C \to C$, such that for arbitrary $c \in C$ and $r \in R$,

(i) $\phi \neq (c\theta)r \subseteq C\theta \Rightarrow cr \neq \phi$,

(ii) $cr \neq \phi \Rightarrow (cr)\theta = (c\theta)r$.

Γ is said to be *uniformly self-isomorphic-embeddable*, if there is a $\{b_0\} \in \mathbb{B}_{\Gamma}$ such that for all $c \in C$ there is a self-isomorphic-embedding θ_c of Γ with $c_0\theta_c = c$.

EXAMPLE 9. Figure 3B is a uniformly self-isomorphic-embeddable rdg.

THEOREM 15. *A uniformly self-isomorphic-embeddable* rdg $\Gamma = (C, R)$ *has a root block.*

Proof. Since Γ is uniformly self-isomorphic-embeddable, there exists a $\{b_0\} \in \mathscr{B}_\Gamma$ defined in Definition 8. For this $\{b_0\}$ and arbitrary $\xi, \eta \in \nabla_R(b_0)$, let $b_0\xi = b_0\eta$. For an arbitrary $c \in C$, by condition (ii) in Definition 8, $(b_0\xi)\theta_c = (b_0\theta_c)\xi = c\xi$, and $(b_0\eta)\theta_c = (b_0\theta_c)\eta = c\eta$. Since c is arbitrary, $\xi = \eta$ is obtained. This completes the proof. ▌

From now on, let the bprdg $\Gamma = (C, R)$ be uniformly self-embeddable, so that there exists a $\{b_0\} \in \mathbb{B}_\Gamma$ such that, for all $c \in C$ there is a self-embedding θ_c of Γ with $c_0\theta_c = c$.

Now, we perform the skeleton mapping on Γ to obtain its skeleton structure $h(\Gamma) = (S, R')$. Let θ_c be a self-embedding of Γ mentioned above, we construct from θ_c, the function θ_c' on S according to the equation

$$\theta_c' = \{\langle c_1\epsilon, c_2\epsilon\rangle \mid \langle c_1, c_2\rangle \in \theta_c\}. \tag{6}$$

The totality of θ_c on C guarantees that θ_c' is a total function on S. Invoking Eq. (6), for any $d \in C$,

$$(d\theta_c)\epsilon = (d\epsilon)\,\theta_c' \tag{7}$$

is obtained.

Figure 3B is a uniformly self-isomorphic-embeddable rdg, but its skeleton structure, depicted in Fig. 4B is not uniformly self-isomorphic-embeddable nor uniformly self-embeddable. We will give a condition that the skeleton structure of Γ, $h(\Gamma) = (S, R')$ is uniformly self-embeddable and $b_0' = b_0\epsilon$ satisfies the condition of uniformly self-embeddability in Definition 7.

LEMMA 16. *For any $d \in C$, and any $c \in C$,*

$$d\theta_c \in c\xi_d,$$

where $\xi_d \in \nabla_R(b_0)$ and $b_0\xi_d = d\gamma$.

Proof. $d \in d\gamma$ and $b_0\xi_d = d\gamma$ imply $d\theta_c \in (b_0\xi_d)\,\theta_c$. By the condition in Definition 7, $(b_0\xi_d)\,\theta_c \subseteq (b_0\theta_c)\,\xi_d$. $b_0\theta_c = c$, therefore, $d\theta_c \in c\xi_d$. ▌

LEMMA 17. *Let θ_s be a self-embedding of $h(\Gamma) = (S, R')$ obeying $b_0'\theta_s = s$. Then, for each $c \in su^{-1}$,*

$$\theta_s = \theta_c'.$$

Here θ_c' comes from Eq. (6).

Proof. For an arbitrary $s_1 \in S$ and an arbitrary $d \in s_1u^{-1}$, let $b_0\xi_d = d\gamma$. Since $d\epsilon = (b_0\xi_d)\epsilon$, $s_1 = (b_0\xi_d)\epsilon = (b_0\epsilon)\,\xi_d' = b_0'\xi_d'$, so that $s_1\theta_s = (b_0'\xi_d')\,\theta_s$. Moreover, by the condition of Definition 7, the functionality of ξ_d' and $b_0'\theta_s = s$, $(b_0'\xi_d')\,\theta_s = (b_0'\theta_s)\,\xi_d' = s\xi_d'$. Therefore, $s_1\theta_s = s\xi_d'$ can be obtained. On the other hand, from Lemma 15, $d\theta_c \in c\xi_d$. And from Eq. (6), $\langle d\epsilon, (c\xi_d)\epsilon\rangle \in \theta_c'$ holds. Hence, $s_1\theta_c' = (d\epsilon)\,\theta_c' = (c\xi_d)\epsilon = (c\epsilon)\,\xi_d' = s\xi_d'$. Since $s_1\theta_s = s\xi_d'$, $s_1\theta_s = s_1\theta_c'$ can be obtained. Here, s_1 is an arbitrary element of S, so $\theta_s = \theta_c'$. ▌

The above lemma shows that if the self-embeddings θ_s of $h(\Gamma)$ obeying $b_0'\theta_s = s$ exist, they are all obtainable from the self-embedding θ_c ($c \in su^{-1}$) according to Eq. (6).

THEOREM 18. *Let* Γ *be uniformly self-embeddable. Then,* $h(\Gamma) = (S, R')$ *is uniformly self-embeddable and* $b_0' = b_0\epsilon$ *satisfies the condition of uniform self-embeddability if and only if* θ_c' *is one-to-one for every* $c \in C$.

Proof. First, let $h(\Gamma)$ be uniformly self-embeddable and for an arbitrary $s \in S$, there exists a self-embedding θ_s satisfying $b_0'\theta_s = s$. Then, by Lemma 16, $\theta_s = \theta_c'$. So the one-to-oneness of θ_s ensures that θ_c' is one-to-one. Since $c \in su^{-1}$ and s is arbitrary, c is also arbitrary on C.

Conversely, for an arbitrary $c \in C$, let θ_c' be one-to-one. The totality of θ_c' is assured. Now for any $s \in S$ and any $r' \in R'$, let $sr' \neq \varnothing$, then for an arbitrary $d \in su^{-1}$, $dr \neq \varnothing$. Since Γ is uniformly self-embeddable, from the condition in Definition 7, $(dr)\theta_c \subseteq (d\theta_c)r \subseteq ((d\theta_c)r)\gamma$. Hence, $((dr)\theta_c)\epsilon = ((d\theta_c)r)\epsilon$.

While, from Eq. (7), $((dr)\theta_c)\epsilon = ((dr)\epsilon)\theta_c' = ((d\epsilon)r')\theta_c' = (d'r')\theta_c' = (sr')\theta_c'$, and $((d\theta_c)r)\epsilon = ((d\theta_c)\epsilon)r' = ((d\epsilon)\theta_c')r' = (d'\theta_c')r' = (s\theta_c')r'$.

Hence, $(sr')\theta_c' = (s\theta_c')r'$ is obtained. Now, it is shown that θ_c' satisfies the condition in Definition 7.

Therefore, $h(\Gamma)$ is uniformly self-embeddable and since $b_0\theta_c = c$, Eq. (7) assures $b'\theta_c' = s$. This completes the proof. ▌

On the preserving of the uniformly self-isomorphic-embeddability under the mapping h, we can immediately conclude that Theorem 18 also holds.

5. REALIZATION OF RDG'S

In the previous sections, the realization problem of an rdg on a memory space has not been discussed. Concerning this problem, the discussion developed in [2] can be extended naturally to our new model.

DEFINITION 9. A *realization* of an rdg $\Gamma = (C, R)$ on the set of addresses A ($\#C \leqslant \#A$) is a pair of mappings,

$$\langle \sigma, \rho \rangle,$$

where $\sigma: C \to A$ is one-to-one and total, $\rho: R^\tau \to \{r_A \mid r_A \subseteq A \times A\}^*$ is a one-to-one monoid homomorphism mapping. Thus, $(1_C)\rho = 1_A$, and for $\xi, \eta \in R^\tau$, $(\xi\eta)\rho = (\xi\rho)(\eta\rho)$. The pair $\langle \sigma, \rho \rangle$ satisfies the following conditions for all $c \in C$ and $r \in R$,

(i) $\varnothing \neq (c\sigma)(r\rho) \subseteq C\sigma \Rightarrow cr \neq \varnothing$,

(ii) $cr \neq \varnothing \Rightarrow (cr)\sigma = (c\sigma)(r\rho)$.

According to this definition, if $\langle \sigma, \rho \rangle$ realizes $\Gamma = (C, R)$, then $(C\sigma, R\rho)$ is isomorphic to Γ.

In [2], a displacement function δ is given as a generalization of the methods of memory assignment, say, a storage mapping function which allocates arrays to storage areas on cores. The class of data graphs which admit δ is specified to be "realizable by relative addressing." Here for the class of bprdg's, "a realization by relative block addressing" is given.

DEFINITION 10. For a bprdg $\mathbf{\Gamma} = (C, R)$, $\langle \sigma, \rho \rangle$ is called a *realization of* $\mathbf{\Gamma}$ *by relative block addressing*, if

(i) there exists a base address $a_0 \in C\sigma$;

(ii) a bijective (= one-to-one onto) displacement function $\delta: \mathscr{B}_\Gamma \to \{\omega \in R^\tau\rho \mid a_0\omega \subseteq C\sigma\} = \Omega$ exists such that for each block $B \in \mathscr{B}_\Gamma$, $B\sigma = a_0(B\delta)$.

Accessing to the block $B\sigma$ on A can be accomplished by knowing the displacement $B\delta$ of the block B and the base address a_0. The following result can be obtained by a similar verification method in [2].

THEOREM 19. *A bprdg* $\mathbf{\Gamma} = (C, R)$ *is realizable by relative block addressing if and only if it admits root blocks.*

Proof. Let A (a set of addresses) exist such that $\#C \leqslant \#A$.

(1) Say that $\mathbf{\Gamma}$ has a root block $\{c_0\}$. Let σ be an arbitrary total one-to-one mapping of C into A. For such σ, define the map $\rho: R^\tau \to \{r_A \mid r_A \subseteq A \times A\}^*$ as follows. For each $\xi \in R^\tau$, $\xi\rho = \sigma^{-1}\xi\sigma$. Then $1_C\rho = \sigma^{-1}1_C\sigma = 1_A$, and for $\xi_1, \xi_2 \in R^\tau$, $(\xi_1\rho)(\xi_2\rho) = (\sigma^{-1}\xi_1\sigma)(\sigma^{-1}\xi_2\sigma) = \sigma^{-1}\xi_1\xi_2\sigma = (\xi_1\xi_2)\rho$. So the ρ specified as above is a monoid homomorphism mapping. First, we show that $\langle \sigma, \rho \rangle$ is a realization of $\mathbf{\Gamma}$ in A. (i) Let $\phi \neq (c\sigma)(r\rho) \subseteq C\sigma$, then $(c\sigma)(r\rho) = (c\sigma)(\sigma^{-1}r\sigma) = (cr)\sigma \subseteq C\sigma$. So $(cr)\sigma \neq \varnothing$, and $cr \neq \varnothing$. (ii) For any $c \in C$ and $r \in R$, let $cr \neq \varnothing$ hence $r \in \nabla_R(c)$. Then $(cr)\sigma = c\sigma\sigma^{-1}r\sigma = (c\sigma)(r\rho)$. Thus, $\langle \sigma, \rho \rangle$ realizes $\mathbf{\Gamma}$ in A. Here we define a total one-to-one function $\beta: \mathscr{B}_\Gamma \to R^\tau$ as follows. For all $B \in \mathscr{B}_\Gamma$, if $c_0\xi = B$, $B\beta = \xi$. Since such ξ is uniquely determined for $B \in \mathscr{B}_\Gamma$, β is a function. It is easily seen that bprdg $\mathbf{\Gamma}$ with root blocks admits such β. This β corresponds to an "addressing scheme" in [2]. Now we show that $\langle \sigma, \rho \rangle$ is an rba-realization. Let β as defined above. Let $a_0 = c_0\sigma$ and $\delta = \beta\rho$. By definition of Ω, $\Omega = (\mathscr{B}\beta)\rho$. For each $B \in \mathscr{B}$, $B\sigma = c_0(B\beta)\sigma = c_0\sigma\sigma^{-1}(B\beta)\sigma = a_0B\beta\rho = a_0(B\delta)$. Thus $\langle \sigma, \rho \rangle$ is an rba-realization.

(2) Conversely, let $\langle \sigma, \rho \rangle$ be an rba-realization of $\mathbf{\Gamma}$ with base address a_0 and displacement function δ. Consider the cell $c_0 = a_0\sigma^{-1} \in C$. Let ξ, η be arbitrary elements of $\nabla(c_0)$ such that $c_0\xi = c_0\eta$. Now, $(c_0\xi)\sigma = (c_0\sigma)(\xi\rho) = a_0(\xi\rho)$, and $(c_0\eta)\sigma = a_0(\eta\rho)$; therefore (i) both $a_0(\xi\rho)$ and $a_0(\eta\rho)$ are included in $C\sigma$, (ii) $a_0(\xi\rho) = a_0(\eta\rho)$. Since δ is onto, from (i), (iii) $^\exists B_1, B_2 \in \mathscr{B}_\Gamma$, $B_1\delta = \xi\rho$, $B_2\delta = \eta\rho$. From our choice of c_0 and the definition of δ, it follows that (iv) $(c_0\xi)\sigma = a_0(\xi\rho) = a_0(B_1\delta) = B_1\sigma$. Since σ is one-to-one, (v) $c_0\xi = B_1$, or equivalently, $(c_0\xi)\delta = \xi\rho$. Similarly, (vi) $c_0\eta = B_2$, or equivalently, $(c_0\eta)\delta = \eta\rho$. Since $c_0\xi = c_0\eta$, and since δ is a function, $\xi\rho = \eta\rho$. Therefore, $\xi = \eta$ since ρ is one-to-one. We have thus shown $c_0 = a_0\sigma^{-1}$ to be a root block cell of $\mathbf{\Gamma}$, and the theorem is proved. ∎

The first half of our proof is a bit different from Rosenberg's.
But Rosenberg's verification method can be also applied validly.

APPENDIX:

Sketch of Proofs of Example 3

Figure 3A

To show {①} is a root block, first we can see the equalities $ab = 1_C$ and $a^2c = a$ hold. $\mathscr{B}_\Gamma = \{B \mid B = ①a^n,\ n = 0, 1, 2,...\}$ $(a^0 = 1_C)$.

Let ξ be an arbitrary relation to the block $①a^n$ $(n = 0, 1, 2,...)$ and let $[\xi]_r$ be the number of occurrences of $r \in R = \{a, b, c\}$ in ξ. Then $[\xi]_a = [\xi]_b + [\xi]_c + n$ must holds. Repeated application of the above equalities into ξ reduces ξ to a^n due to $[\xi]_a = [\xi]_b + [\xi]_c + n$.

For example, consider the block $①a = \{③, ④\}$. Let $\xi = a^3bc$, so $①a = ①\xi$. Then

$$\begin{aligned} a^3bc &= a^2(ab)c \\ &= a^2c && (ab = 1_C), \\ &= a \quad \because && (a^2c = a). \end{aligned}$$

In this way, every relation to the block $①a^n$ $(n = 0, 1, 2,...)$ can be shown to be equal (as a set of binary relations) to a^n, so all are equal. Since, $①a^n$ is an arbitrary block, {①} is a root block. ▮

Figure 3B

First, to show {①} is a root block, note that the equalities $ab = c^2 = 1_C$, $ac = a$ hold. $\mathscr{B}_\Gamma = \{B \mid B = ①a^n,\ n = 0, 1, 2,...\} \cup \{B \mid B = ①ca^n,\ n = 0, 1,...\}$.

Let ξ be an arbitrary relation to the block $①a^n$. For such ξ, $[\xi]_a = [\xi]_b + n$ must hold. Again, repeated application of the above equalities reduces ξ to a^n, due to $[\xi]_a = [\xi]_b + n$.

For example, consider the block $①a = \{③, ④\}$. Let $\xi = a^3bcb$, so $①a = ①\xi$. Then,

$$\begin{aligned} a^3bcb &= a^2(ab)cb \\ &= a^2cb && (ab = 1), \\ &= a(ac)b \\ &= a^2b && (ac = a), \\ &= a(ab) \quad \because \\ &= a && (ab = 1). \end{aligned}$$

Similarly every relation to the block $①ca^n$ can be shown to be equal to ca^n. That {①} is a root block now follows. By a similar method, {②} is also a root block. ▮

Proof of Uniformly Self-Embeddability of Fig. 7

Since Fig. 7 is obviously isomorphic to Fig. 8, we prove on Fig. 8 (where ϵ denotes a null string). First, for $\mathbf{k} \in C - \{*\}$, let $\theta_k^1 = \{\langle *, \mathbf{k}\rangle\} \cup \{\langle n, \mathbf{k0n}\rangle \mid \mathbf{n} \in C - \{*\}\}$, here $\mathbf{k0n}$ denotes the concatenation of strings $\mathbf{k}$, 0, $\mathbf{n}$. $\theta_{\mathbf{k}}^1$ is a self-embedding of Fig. 8. To wit, first $\theta_{\mathbf{k}}^1$ is a total injection, and

$$(*a)\,\theta_{\mathbf{k}}^1 = \{\epsilon\}\,\theta_{\mathbf{k}}^1 = \{\mathbf{k0}\}, \qquad (*\theta_{\mathbf{k}}^1)a = \mathbf{k}a = \{\mathbf{k0}, \mathbf{k1}\}, \qquad \text{so } (*a)\,\theta_{\mathbf{k}}^1 \subseteq (*\theta_{\mathbf{k}}^1)a.$$

For $\mathbf{n} \in C - \{*\}$,

$$(\mathbf{n}a)\,\theta_{\mathbf{k}}^1 = \{\mathbf{n0}, \mathbf{n1}\}\,\theta_{\mathbf{k}}^1 = (\mathbf{k\,0n0}, \mathbf{k0n1}\}, \qquad (\mathbf{n}\theta_{\mathbf{k}}^1)a = (\mathbf{k0n})a = \{\mathbf{k0n0}, \mathbf{k0n1}\},$$

so

$$(\mathbf{n}a)\,\theta_{\mathbf{k}}^1 = (\mathbf{n}\theta_{\mathbf{k}}^1)a.$$

And $(\epsilon b)\,\theta_{\mathbf{k}}^1 = \{*\}\,\theta_{\mathbf{k}}^1 = \{\mathbf{k}\}$, $(\epsilon\theta_{\mathbf{k}}^1)b = \{\mathbf{k0}\}b = \{\mathbf{k}\}$, so $(\epsilon b)\,\theta_{\mathbf{k}}^1 = (\epsilon\theta_{\mathbf{k}}^1)b$.

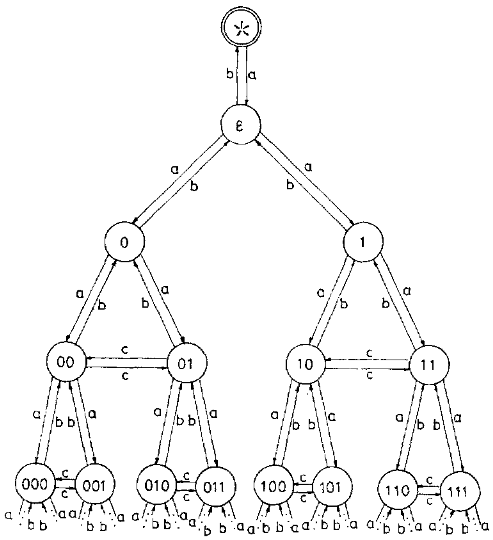


Fig. 8. An rdg isomorphic to Fig. 7.

For $\mathbf{n0} \in C - \{*\}$, $((\mathbf{n0})b)\,\theta_{\mathbf{k}}^1 = \{\mathbf{n}\}\,\theta_{\mathbf{k}}^1 = \{\mathbf{k0n}\}$, $((\mathbf{n0})\,\theta_{\mathbf{k}}^1)b = (\mathbf{k0n0})b = \{\mathbf{k0n}\}$, hence $((\mathbf{n0})b)\,\theta_{\mathbf{k}}^1 = ((\mathbf{n0})\,\theta_{\mathbf{k}}^1)b$. Similarly for $\mathbf{n1} \in C - \{*\}$, $((\mathbf{n1})b)\,\theta_{\mathbf{k}}^1 = ((\mathbf{n1})\,\theta_{\mathbf{k}}^1)b$.

Last, for $\mathbf{n1} \in C - \{*, \epsilon, 0, 1\}$, $((\mathbf{n0})c)\,\theta_{\mathbf{k}}^1 = \{\mathbf{n1}\}\,\theta_{\mathbf{k}}^1 = \{\mathbf{k0n1}\}$, $((\mathbf{n0})\,\theta_{\mathbf{k}}^1)c = (\mathbf{k0n0})c = \{\mathbf{k0n1}\}$, so that $((\mathbf{n0})c)\,\theta_{\mathbf{k}}^1 = ((\mathbf{n0})\,\theta_{\mathbf{k}}^1)c$. Similarly for $\mathbf{n1} \in C - \{*, \epsilon, 0, 1\}$, $((\mathbf{n1})c)\,\theta_{\mathbf{k}}^1 = ((\mathbf{n1})\,\theta_{\mathbf{k}}^1)c$.

In like manner, $\theta_{\mathbf{k}}^2 = \{\langle *, \mathbf{k}\rangle\} \cup \{\langle \mathbf{n}, \mathbf{k1n}\rangle \mid \mathbf{n} \in C - \{*\}\}$ is also a self-embedding. For $*$, let $\theta_* = 1_C$ (obviously a self-embedding). Thus Fig. 8 (Fig. 7) is uniformly self-embeddable. ▮

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